



ST-6 Statistical Weights on Reserve Estimates

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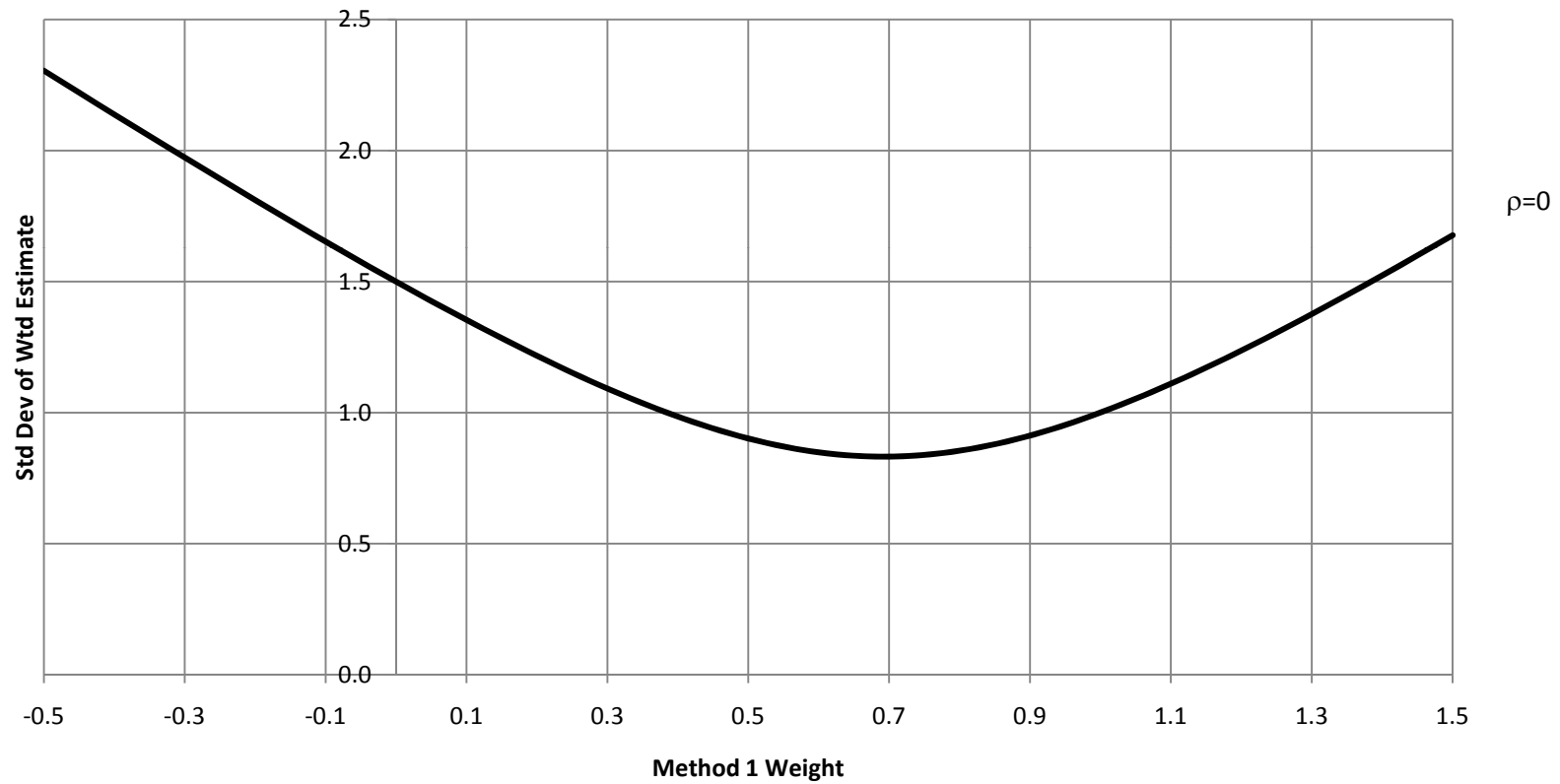
Competing Reserve Estimates

- Why do we use multiple methods to estimate reserves?
- How do we typically use multiple reserve estimates together?
- What should be the drivers of the decisions to rely more on particular estimates versus others.

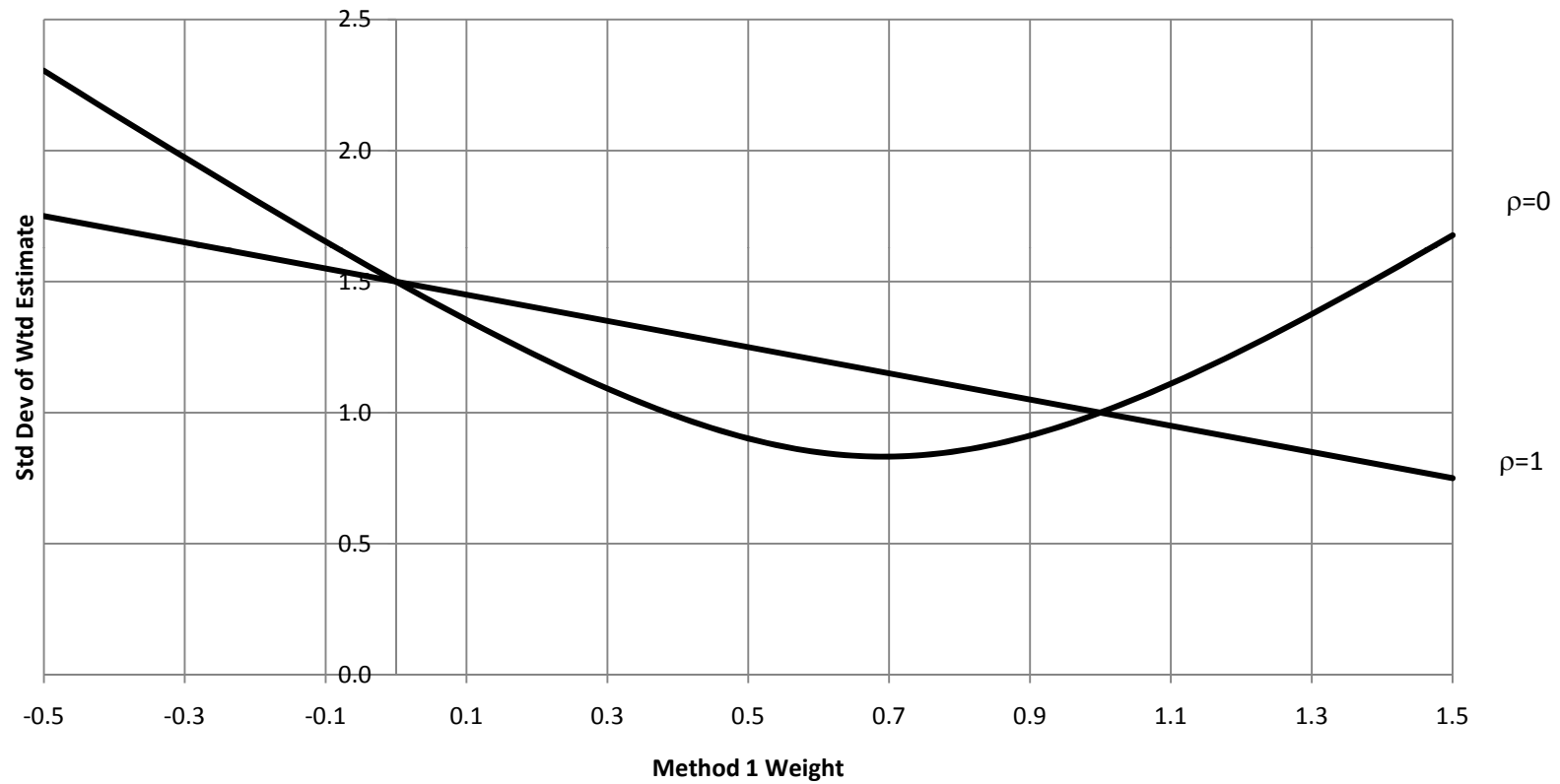
Basic Example- Two Estimates

- Two estimates R_1 and R_2 and of the same unknown quantity X
- Both estimates ultimately will be in error. The error is currently unknown
 - $R_1 = X + \varepsilon_1, R_2 = X + \varepsilon_2$
- Both estimates are unbiased
 - $E(\varepsilon_1)=0, E(\varepsilon_2)=0$
- Each estimate has its own variance
 - $\text{Var}(\varepsilon_1)=\sigma_1^2, \text{Var}(\varepsilon_2)=\sigma_2^2$
- The two estimates errors are correlated with correlation coefficient ρ
- A combined estimate
 - $R_c = w_1R_1 + w_2R_2$ where $w_1 + w_2 = 1$
 - $R_c = X + \varepsilon_c$
 - $\varepsilon_c = w_1\varepsilon_1 + w_2\varepsilon_2$
- $E(\varepsilon_c)=0$
- $\text{Var}(\varepsilon_c) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2\rho w_1w_2\sigma_1\sigma_2$
- $\sigma_1=1, \sigma_2=1.5$

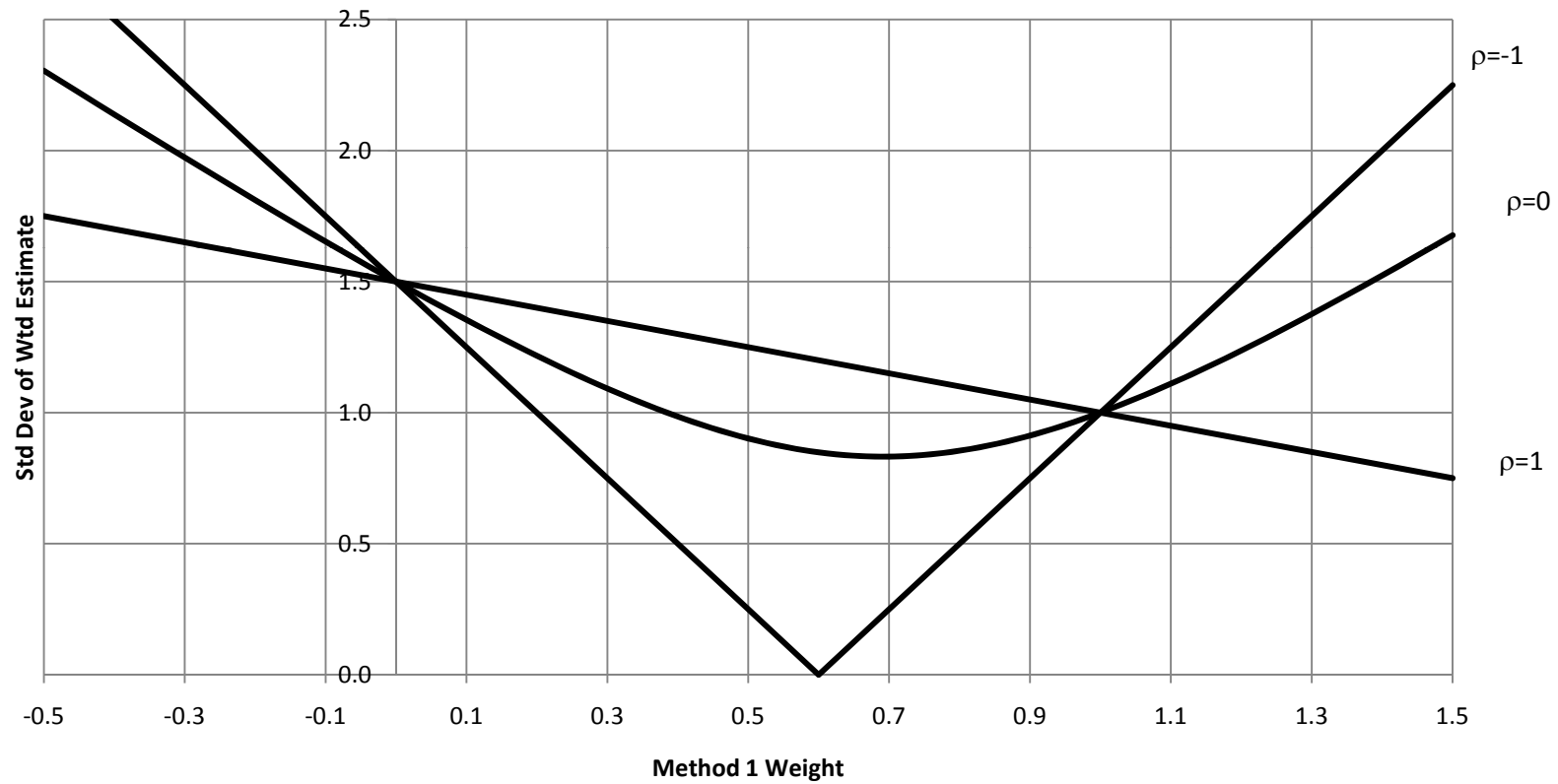
Independent Estimates



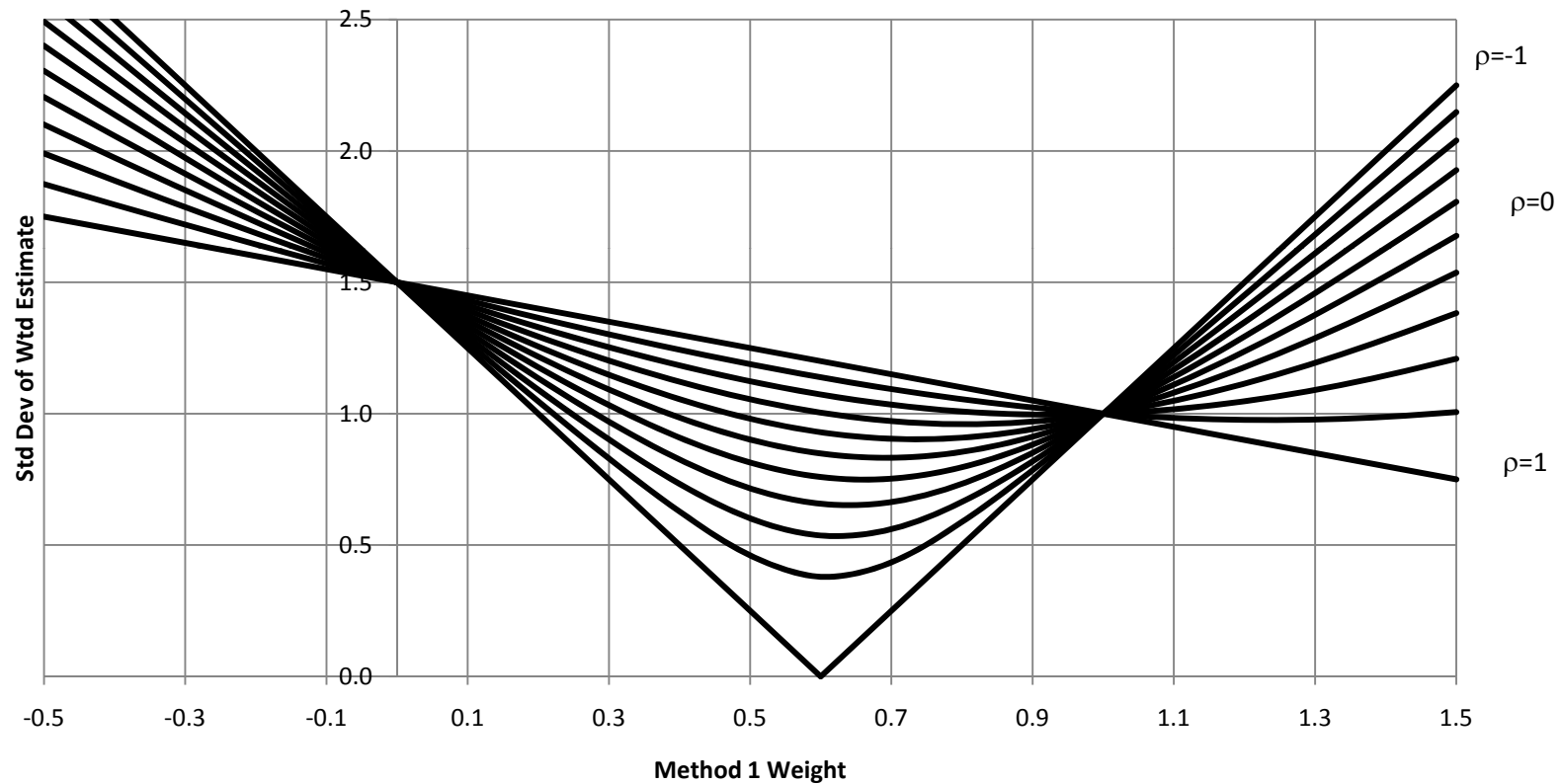
Correlation = 1



Correlation = -1



Importance of Correlation



Optimal (Minimum Variance) Weights

- Optimal weights are given by:

$$w_1 = (\sigma_2^2 - \rho\sigma_1\sigma_2) / (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$$

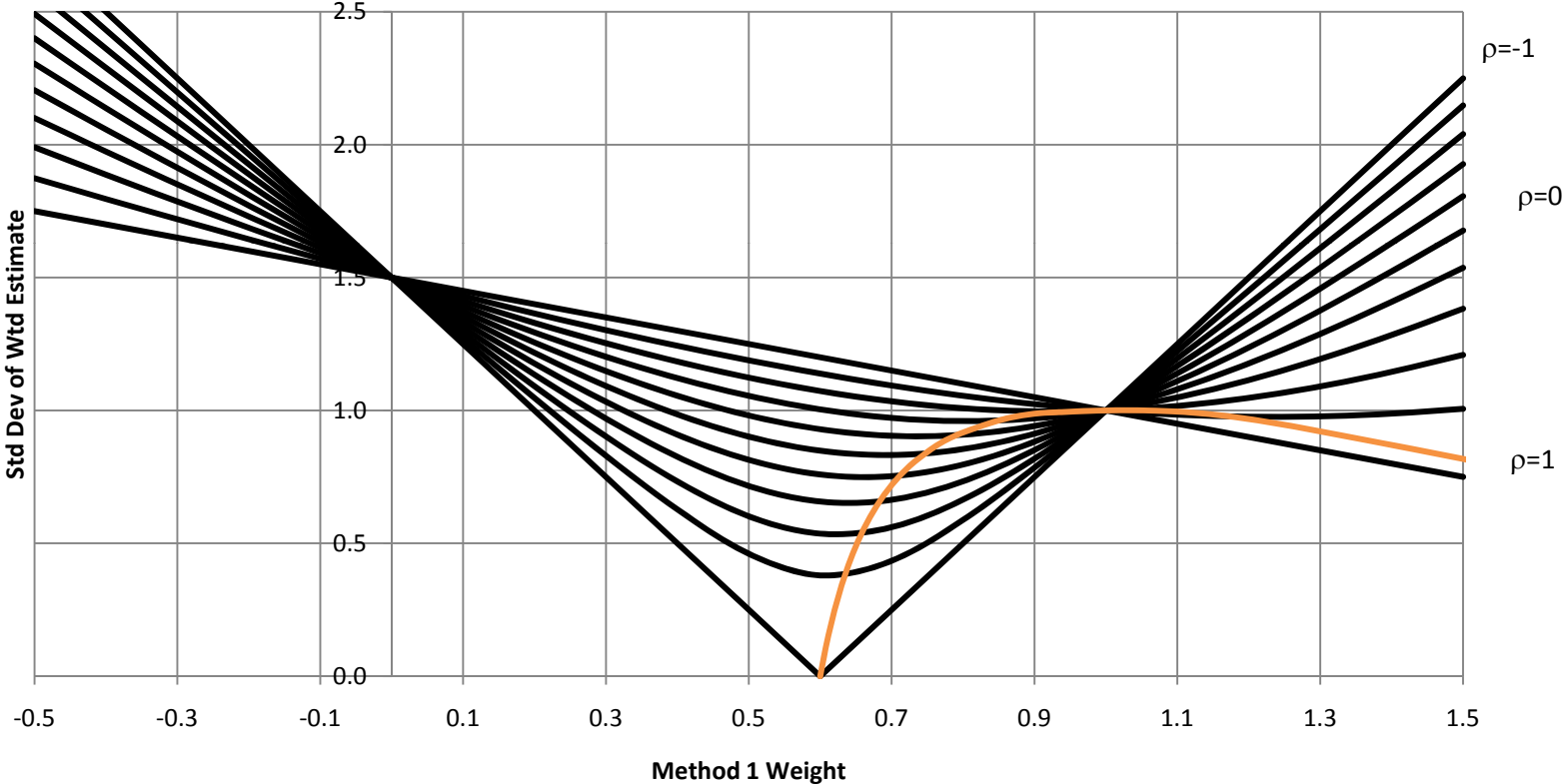
$$w_2 = (\sigma_1^2 - \rho\sigma_1\sigma_2) / (\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$$

- Special case of independence

$$w_1 = \sigma_2^2 / (\sigma_1^2 + \sigma_2^2) = \sigma_1^{-2} / (\sigma_2^{-2} + \sigma_1^{-2})$$

(i.e. weight inverse to variance)

Optimal Weight Given Correlation



Generalized Method

- Let \mathbf{C} denote a n by n covariance matrix describing the error potential of n different estimates of an unknown amount.
- Let \mathbf{w} denote an n dimensional vector of weights adding to one to apply to the estimates to arrive at a combined estimate with minimum variance.
- The elements of \mathbf{w} are proportional to the corresponding sums of row (or column) vectors of \mathbf{C}^{-1} .
- The variance of the combined estimate is given by $1/(\text{sum of all the elements of } \mathbf{C}^{-1})$.

Examples

σ_1	σ_2	ρ
1	1.5	0

C

1	0
0	2.25

C⁻¹

1	0
0	0.444444

Sum of
Row
Vectors

1	0.692308	w_1
0.444444	0.307692	w_2

Total 1.444444 0.692308 Combined variance

0.83205 Combined SD

Examples

σ_1	σ_2	ρ
1	1.5	0.5

C

1	0.75
0.75	2.25

C⁻¹

1.333333	-0.444444
-0.444444	0.592593

Sum of
Row
Vectors

0.888889	0.857143	w_1
0.148148	0.142857	w_2

Total 1.037037 0.964286 Combined variance

0.981981 Combined SD

Examples

σ_1	σ_2	ρ
1	1.5	0.9

C

1	1.35
1.35	2.25

C⁻¹

5.263158	-3.15789
-3.15789	2.339181

Sum of
Row
Vectors

2.105263	1.636364	w ₁
-0.81871	-0.63636	w ₂

Total 1.28655 0.777273 Combined variance

0.881631 Combined SD

Reserving Examples



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CREATIVE SOLUTIONS TO COMPLEX PROBLEMS

Reserving Example 1

Large Specialty Writer - Other Liability Occurrence

	Paid LDF	Incurred LDF	Paid BF	Incurred BF	St. Dev of Reserve
Paid LDF	1.000	0.281	0.597	0.209	272,019
Incurred LDF	0.281	1.000	0.117	0.700	125,348
Paid BF	0.597	0.117	1.000	0.344	104,630
Incurred BF	0.209	0.700	0.344	1.000	134,155

Reserving Example 1

Paid LDF Residuals									
	Age2	Age3	Age4	Age5	Age6	Age7	Age8	Age9	Age10
▶ 1999	17,912	-11,896	16,457	-9,943	6,222	-18,266	17,149	1,804	-13
2000	44,587	-12,791	7,884	13,106	2,395	-11,426	-14,315	-1,436	
2001	18,259	-32,745	-19,191	21,969	-6,504	32,050	-3,260		
2002	-4,648	-50,777	-4,652	-4,257	-1,931	-1,925			
2003	-46,789	-22,850	6,031	-7,909	282				
2004	-36,029	37,577	-17,190	-12,407					
2005	24,216	17,720	10,593						
2006	20,368	18,236							
2007	38,196								
2008									

Incurred LDF Residuals									
	Age2	Age3	Age4	Age5	Age6	Age7	Age8	Age9	Age10
▶ 1999	39,570	29,488	-7,316	-121	-21,518	8,114	22,967	416	214
2000	24,925	35,436	7,060	-4,026	5,198	-19,120	-4,703	-410	
2001	-11,103	-10,880	8,273	19,660	23,714	22,035	-18,369		
2002	4,156	-22,603	15,404	-14,417	-4,263	-10,633			
2003	-45,257	-40,145	19,805	-8,197	-2,443				
2004	-24,270	13,656	-17,535	7,369					
2005	55,623	14,704	-25,048						
2006	-45,808	-20,196							
2007	1,941								
2008									

Paid Bornhuetter-Ferguson Residuals									
	Age2	Age3	Age4	Age5	Age6	Age7	Age8	Age9	Age10
▶ 1999	2,760	-13,268	11,361	-10,817	4,764	-19,552	14,796	1,217	-7
2000	17,513	-4,357	9,737	16,303	5,859	-7,416	-12,048	-878	
2001	37,886	4,825	-2,622	31,803	1,925	40,794	6,059		
2002	5,816	-43,181	-16,448	-15,380	-9,012	-10,524			
2003	1,300	-9,869	7,799	-5,218	1,209				
2004	-10,555	34,731	-6,408	-7,106					
2005	2,370	14,355	14,180						
2006	-8,866	3,341							
2007	9,099								
2008									

Incurred Bornhuetter-Ferguson Residuals									
	Age2	Age3	Age4	Age5	Age6	Age7	Age8	Age9	Age10
▶ 1999	3,734	23,145	-6,235	161	-21,754	7,269	22,169	-52	624
2000	4,136	32,516	11,379	-1,110	7,311	-17,358	-2,593	-899	
2001	18,009	2,377	14,709	23,705	28,497	26,308	-8,300		
2002	-14,691	-32,331	5,187	-18,151	-9,141	-13,884			
2003	-3,037	-32,115	17,745	-7,443	-2,785				
2004	-9,506	13,185	-15,526	7,305					
2005	22,871	16,383	-21,495						
2006	-16,473	-20,127							
2007	12,234								
2008									

Build the Covariance Matrix

		Paid LDF	Incurred LDF	Paid BF	Incurred BF
		272,019	125,348	104,630	134,155
Paid LDF	272,019	1.000	0.281	0.597	0.209
Incurred LDF	125,348	0.281	1.000	0.117	0.700
Paid BF	104,630	0.597	0.117	1.000	0.344
Incurred BF	134,155	0.209	0.700	0.344	1.000

Covariance Matrix

		Paid LDF	Incurred LDF	Paid BF	Incurred BF
Paid LDF		7.40E+10	9.58E+09	1.70E+10	7.63E+09
Incurred LDF		9.58E+09	1.57E+10	1.53E+09	1.18E+10
Paid BF		1.70E+10	1.53E+09	1.09E+10	4.83E+09
Incurred BF		7.63E+09	1.18E+10	4.83E+09	1.80E+10

Invert and Add

Inverted Covariance Matrix					Row Total	Ratio to Grand Total
	2.43E-11	-2.23E-11	-4.14E-11	1.54E-11	-2.40E-11	-0.147
	-2.23E-11	1.50E-10	5.98E-11	-1.04E-10	8.27E-11	0.507
	-4.14E-11	5.98E-11	1.78E-10	-6.92E-11	1.27E-10	0.778
	1.54E-11	-1.04E-10	-6.92E-11	1.36E-10	-2.24E-11	-0.137
Column Total	-2.40E-11	8.27E-11	1.27E-10	-2.24E-11	1.63E-10	

Combined SD = 78,261

Weights

Paid LDF	-14.7%
Incurred LDF	50.7%
Paid BF	77.8%
Incurred BF	-13.7%

Constrained Solution

Covariance Matrix						
	Paid LDF	Incurred LDF	Paid BF	Incurred BF		
Paid LDF	7.40E+10	9.58E+09	1.70E+10	7.63E+09		
Incurred LDF	9.58E+09	1.57E+10	1.53E+09	1.18E+10		
Paid BF	1.70E+10	1.53E+09	1.09E+10	4.83E+09		
Incurred BF	7.63E+09	1.18E+10	4.83E+09	1.80E+10		

Inverted Covariance Matrix				Row Total	Ratio to Grand Total
	1.29E-10	2.17E-11	-9.03E-11	6.06E-11	0.435
	2.17E-11	1.07E-10	-4.30E-11	8.60E-11	0.617
	-9.03E-11	-4.30E-11	1.26E-10	-7.15E-12	-0.051
Column Total	6.06E-11	8.60E-11	-7.15E-12	1.39E-10	

Combined SD = 84,680

Weights

Incurred LDF	43.5%
Paid BF	61.7%
Incurred BF	-5.1%

Constrained Solution

Covariance Matrix

	Paid LDF	Incurred LDF	Paid BF	Incurred BF
Paid LDF	7.40E+10	9.58E+09	1.70E+10	7.63E+09
Incurred LDF	9.58E+09	1.57E+10	1.53E+09	1.18E+10
Paid BF	1.70E+10	1.53E+09	1.09E+10	4.83E+09
Incurred BF	7.63E+09	1.18E+10	4.83E+09	1.80E+10

Inverted Covariance Matrix				Row Total	Ratio to Grand Total
	2.10E-11	-3.25E-11	-1.75E-13	-1.17E-11	-0.099
	-3.25E-11	1.54E-10	-2.75E-11	9.39E-11	0.799
	-1.75E-13	-2.75E-11	6.30E-11	3.53E-11	0.301
Column Total	-1.17E-11	9.39E-11	3.53E-11	1.18E-10	

Combined SD = 92,236

Weights

Paid LDF	-9.9%
Paid BF	79.9%
Incurred BF	30.1%

Constrained Solution

- Systematically eliminate indications
- With four methods there are 15 different possible combinations:
 - 1 that includes all four
 - 4 that include three indications
 - 6 that include two indications
 - 4 that include one indication
- To find constrained solution, select the one of these with the lowest standard deviation from among those with no negative weights.

Reserving Example 1

Large Specialty Writer - Other Liability Occurrence

	Paid LDF	Incurred LDF	Paid BF	Incurred BF	St. Dev of Reserve
Paid LDF	1.000	0.281	0.597	0.209	272,019
Incurred LDF	0.281	1.000	0.117	0.700	125,348
Paid BF	0.597	0.117	1.000	0.344	104,630
Incurred BF	0.209	0.700	0.344	1.000	134,155
Optimal Weights	-0.1474	0.5066	0.7778	-0.137	78,261
Constrained Optimal Weights	0	0.3991	0.6009	0	84,790

Reserving Example 1

Hindsight testing of 2008 estimates by 2009 revision

Large Specialty Lines Writer

Other Liability Occurrence

Accident Year	Paid LDF	Incurred		Incurred B-F	Optimal Weighting	Constrained	Even Weighting
		LDF	Paid B-F			Optimal Weighting	
2000	5,010	2,720	5,331	3,352	4,327	4,289	4,103
2001	-10,403	18,080	-5,761	16,486	3,953	3,754	4,601
2002	-14,918	-15,710	-20,782	-22,424	-18,852	-18,758	-18,459
2003	-37,503	-15,652	-31,760	-14,375	-25,135	-25,331	-24,823
2004	-20,469	-9,601	-13,989	-8,531	-11,559	-12,238	-13,148
2005	-45,371	-14,448	-26,897	-11,659	-19,955	-21,929	-24,594
2006	25,097	-4,732	9,799	-7,117	2,500	4,000	5,762
2007	-2,795	23,820	-558	22,470	8,967	9,171	10,734
2008	-116,698	8,986	-12,492	-6,955	12,990	-3,920	-31,790
	-218,050	-6,537	-97,109	-28,753	-42,763	-60,962	-87,612

All development factors and seed loss ratios were left unchanged

Bold cells indicate the most accurate in hindsight

Reserving Example 2

Regional Carrier - Commercial Multi-Peril

	Paid LDF	Incurred LDF	Paid BF	Incurred BF	St. Dev of Reserve
Paid LDF	1.000	0.178	0.518	0.172	63,700
Incurred LDF	0.178	1.000	0.098	0.941	38,190
Paid BF	0.518	0.098	1.000	0.258	42,589
Incurred BF	0.172	0.941	0.258	1.000	36,653

Reserving Example 2

Paid LDF Residuals

	Age2	Age3	Age4	Age5	Age6	Age7	Age8	Age9	Age10
▶ 1999	-5,823	-5,727	-1,735	-7,045	-3,799	-106	-1,347	294	13
2000	2,532	-3,524	1,542	-4,305	-1,440	-1,447	857	-88	
2001	-302	-5,467	-2,207	-3,196	-2,403	832	574		
2002	4,390	3,322	3,025	1,643	2,672	1,231			
2003	-2,279	1,392	8,587	7,121	5,156				
2004	-2,503	-8,188	-4,211	-1,258					
2005	11,426	22,688	-4,747						
2006	5,292	-4,821							
2007	-2,183								
2008									

Incurred LDF Residuals

	Age2	Age3	Age4	Age5	Age6	Age7	Age8	Age9	Age10
▶ 1999	-17,531	955	1,342	-1,386	646	-469	-419	1,017	80
2000	1,552	4,066	8,489	-969	1,630	-1,252	722	-1,132	
2001	-1,531	-762	-6,411	-3,835	1,426	1,127	153		
2002	-1,150	6,311	-169	3,763	-2,634	289			
2003	-518	-1,362	11,792	4,036	-1,533				
2004	19,074	2,017	-7,373	-2,199					
2005	25,629	-6,896	-7,217						
2006	-14,964	-4,782							
2007	-10,688								
2008									

Paid Bornhuetter-Ferguson Residuals

	Age2	Age3	Age4	Age5	Age6	Age7	Age8	Age9	Age10
▶ 1999	3,153	-2,763	18	-5,805	-3,429	102	-1,055	307	31
2000	17,514	3,255	6,781	-103	780	-314	1,694	126	
2001	9,557	-1,314	561	-1,200	-1,440	1,367	1,103		
2002	-2,669	892	1,124	415	1,936	1,075			
2003	-11,734	-3,057	4,740	4,810	4,010				
2004	-1,734	-8,277	-5,404	-2,490					
2005	1,096	19,885	-4,877						
2006	-16,327	-10,405							
2007	-15,882								
2008									

Incurred Bornhuetter-Ferguson Residuals

	Age2	Age3	Age4	Age5	Age6	Age7	Age8	Age9	Age10
▶ 1999	-15,262	1,567	1,558	-1,359	641	-483	-413	1,029	95
2000	5,809	7,236	9,341	-732	1,570	-1,374	731	-1,012	
2001	2,104	1,844	-5,359	-3,777	1,395	1,071	132		
2002	-2,593	5,734	-369	3,720	-2,626	306			
2003	-3,177	-2,653	11,347	3,961	-1,520				
2004	15,774	1,633	-7,319	-2,280					
2005	23,063	-6,336	-7,298						
2006	-18,312	-7,361							
2007	-13,464								
2008									

Reserving Example 2

Regional Carrier - Commercial Multi-Peril

	Paid LDF	Incurred LDF	Paid BF	Incurred BF	St. Dev of Reserve
Paid LDF	1.000	0.178	0.518	0.172	63,700
Incurred LDF	0.178	1.000	0.098	0.941	38,190
Paid BF	0.518	0.098	1.000	0.258	42,589
Incurred BF	0.172	0.941	0.258	1.000	36,653
Optimal Weights	-0.0072	0.7793	0.478	-0.2501	29,668
Constrained Optimal Weights	0.0089	0.5581	0.433	0	29,779

Reserving Example 2

Hindsight testing of 2008 estimates by 2009 revision

Regional Carrier

Commercial Multi-Peril

Accident Year	Paid LDF	Incurred		Incurred B-F	Optimal Weighting	Constrained	Even Weighting
		LDF	Paid B-F			Optimal Weighting	
2000	-245	-361	-31	-242	-234	-217	-220
2001	1,712	306	1,775	360	985	955	1,038
2002	87	-386	185	-395	-114	-135	-127
2003	9,069	8,294	8,406	8,280	8,345	8,349	8,512
2004	-2,588	2,596	-3,285	2,609	-181	3	-167
2005	-11,551	-7,348	-10,970	-7,387	-9,039	-8,954	-9,314
2006	-1,882	-6,260	-7,097	-7,741	-6,321	-6,583	-5,745
2007	-9,700	-5,892	-13,815	-7,750	-9,187	-9,357	-9,289
2008	-34,077	-25,856	8,939	-11,796	-12,681	-10,863	-15,698
	-49,175	-34,907	-15,893	-24,062	-28,428	-26,801	-31,009

All development factors and seed loss ratios were left unchanged

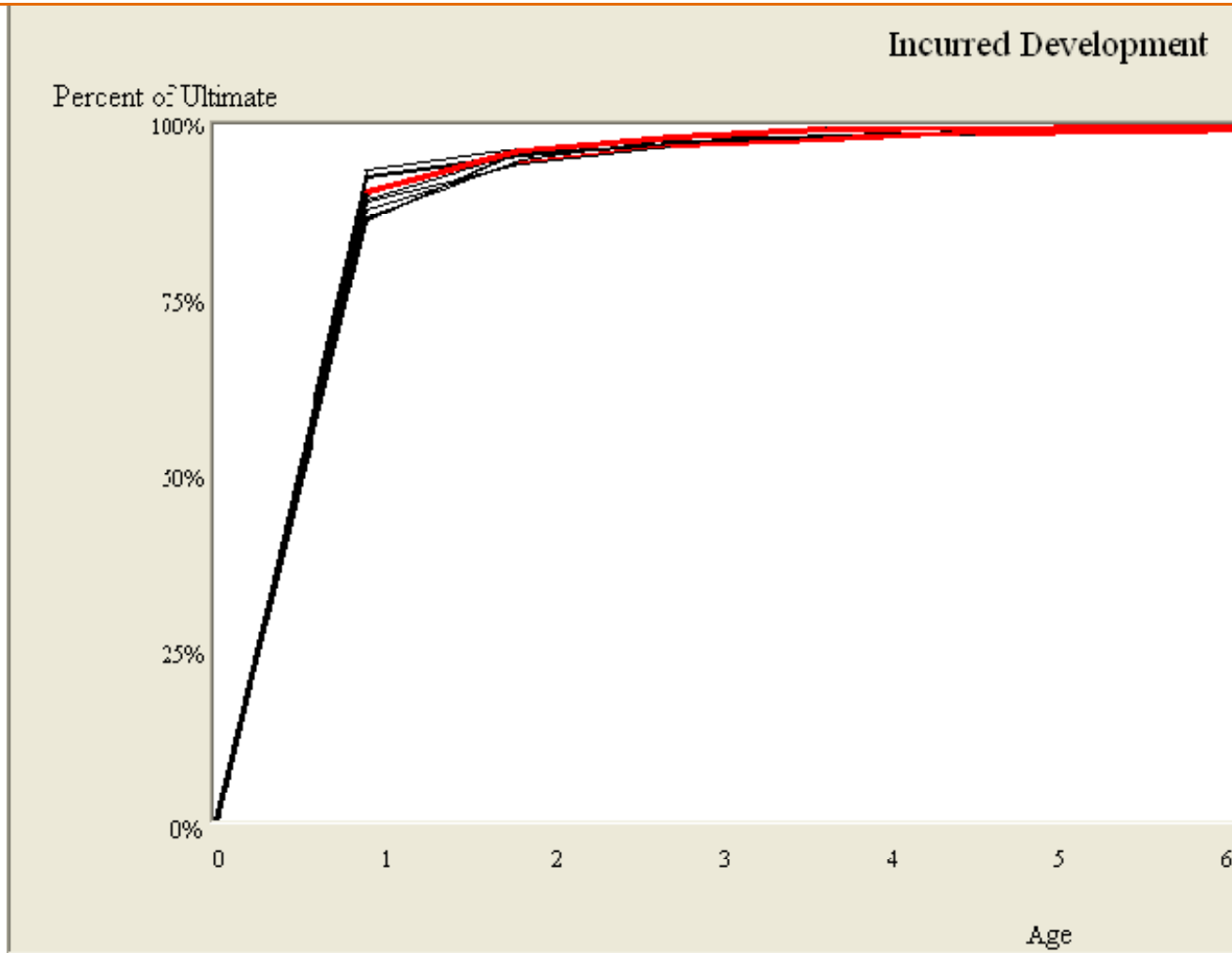
Bold cells indicate the most accurate in hindsight

Reserving Example 3

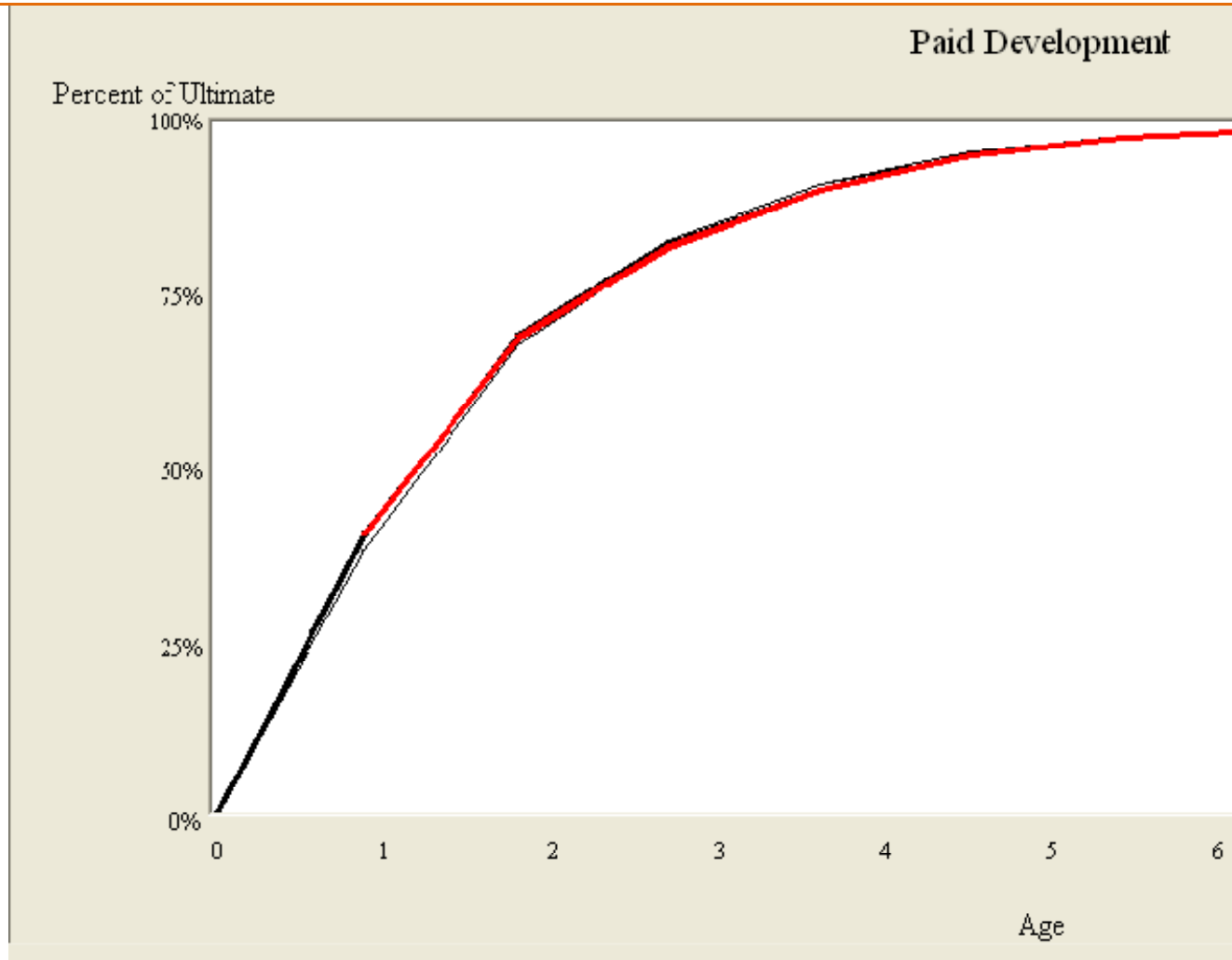
Large Personal Lines Carrier - Private Passenger Auto Liability

	Paid LDF	Incurred LDF	Paid BF	Incurred BF	St. Dev of Reserve
Paid LDF	1.000	0.738	0.748	0.720	224,790
Incurred LDF	0.738	1.000	0.774	0.992	315,570
Paid BF	0.748	0.774	1.000	0.780	219,490
Incurred BF	0.720	0.992	0.780	1.000	393,281
Optimal Weights	0.1187	2.386	0.5514	-2.0561	142,567
Constrained Optimal Weights	0.4527	0	0.5473	0	207,537

Reserving Example 3



Reserving Example 3



Reserving Example 3

Hindsight testing of 2008 estimates by 2009 revision

Large Personal Lines Carrier

Private Passenger Auto Liability

Accident Year	Paid LDF	Incurred		Incurred B-F	Optimal Weighting	Constrained	Even Weighting
		LDF	Paid B-F			Optimal Weighting	
2000	1,025	15,841	1,084	15,559	6,525	1,057	8,377
2001	13,276	-2,921	13,524	-2,772	7,763	13,412	5,277
2002	-918	1,235	-1,197	1,323	-543	-1,071	111
2003	4,859	-1,847	6,575	-2,036	3,981	5,798	1,888
2004	8,967	3,874	1,314	3,548	3,737	4,779	4,426
2005	7,062	30,986	7,497	30,664	15,856	7,300	19,052
2006	-4,200	17,321	-6,732	14,685	6,924	-5,586	5,269
2007	-28,380	59,855	-1,263	60,395	14,571	-13,539	22,652
2008	205,257	150,938	111,679	137,597	163,169	154,042	151,368
Total	206,948	275,282	132,481	258,963	221,984	166,192	218,419

All development factors and seed loss ratios were left unchanged

Bold cells indicate the most accurate in hindsight

Reserving Examples Summary

Hindsight Testing of 2008 by 2009

Total Change to Comparable Ultimate Loss Estimates

	Large Specialty Oth Liab Occ	Regional Carrier CMP	Large Personal Lines Priv Pass AL
Paid LDF	(218,050)	(49,175)	206,948
Incurred LDF	(6,537)	(34,907)	275,282
Paid B-F	(97,109)	(15,893)	132,481
Incurred B-F	(28,753)	(24,062)	258,963
Optimal Weighting	(42,763)	(28,428)	221,984
Constrained Optimal Weighting	(60,962)	(26,801)	166,192
Even Weighting	(87,612)	(31,009)	218,419

Conclusion

- In considering competing estimates of reserves, a goal should be to improve accuracy (reduce uncertainty)
- All of the following should be considerations when looking at competing estimates
 - Bias
 - Stand-alone variability
 - Correlation